Lecture 2: Conceptual Algorithm (1)

The SIMPLE Algorithm
An introduction to solving incompressible flow problems

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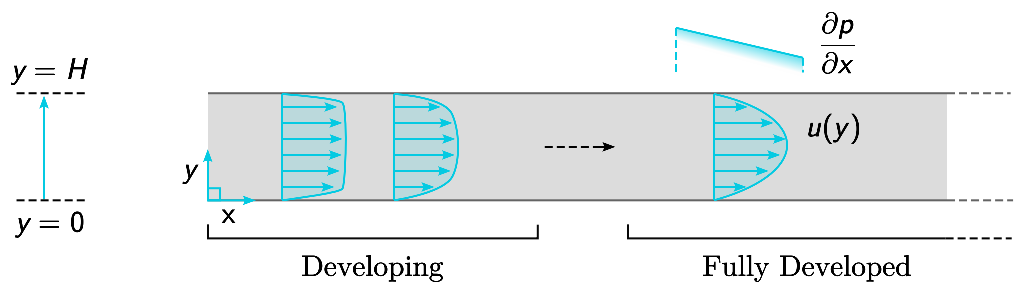
Before deriving the equations that are used in the SIMPLE algorithm, a *conceptual algorithm* is developed first. This conceptual algorithm has all of the key features of the SIMPLE algorithm and is useful for building an understanding of the algorithm first, before the detailed equations are presented in future lectures.

1 Hagen-Poiseuille Flow

Consider fully-developed laminar flow between two infinitely wide parallel plates. When the flow is fully developed (far from the entrance), Hagen and Poiseuille [2] showed that the velocity profile is:

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(yH - y^2\right) \tag{1}$$

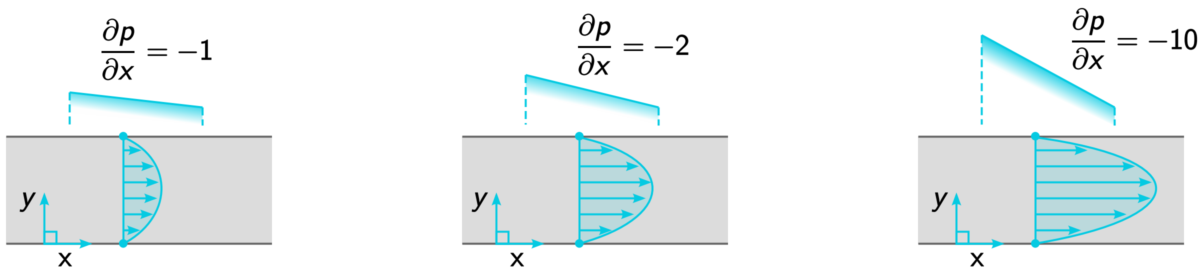
where u is the axial velocity, μ is the dynamic viscosity, $\partial p/\partial x$ is the pressure gradient, H is the height of the channel, x is the axial coordinate and y is the vertical coordinate. The Figure below shows a diagram of the geometry and coordinate system for Hagen–Poiseuille flow.



The Hagen-Poiseuille equation shows that:

1. The axial velocity (u) is proportional to the (negative) pressure gradient ☐ True ☐ False
2. The axial velocity (u) is negative ☐ True ☐ False
3. A negative sign is required because the pressure decreases in the x direction to drive the flow along the channel ☐ True ☐ False
4. The shape of the velocity profile is quadratic, with a maximum at the channel centreline ☐ True ☐ False

If the magnitude of the pressure gradient is increased (the pressure gradient is more negative), u increases and the shape of the velocity profile changes.



It follows that different pressure gradients result in different velocity profiles. We are going to use this idea to develop the central idea of the conceptual algorithm. Applying different pressure gradients results in different velocity profiles.

2 Central Idea

The central idea of the conceptual algorithm and the SIMPLE algorithm is to treat each momentum equation as a *single-input single-output system* (a 'black box').

Input $\frac{\partial p}{\partial x}$

↓

Solve u Momentum Equation

$$\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right)$$

↓

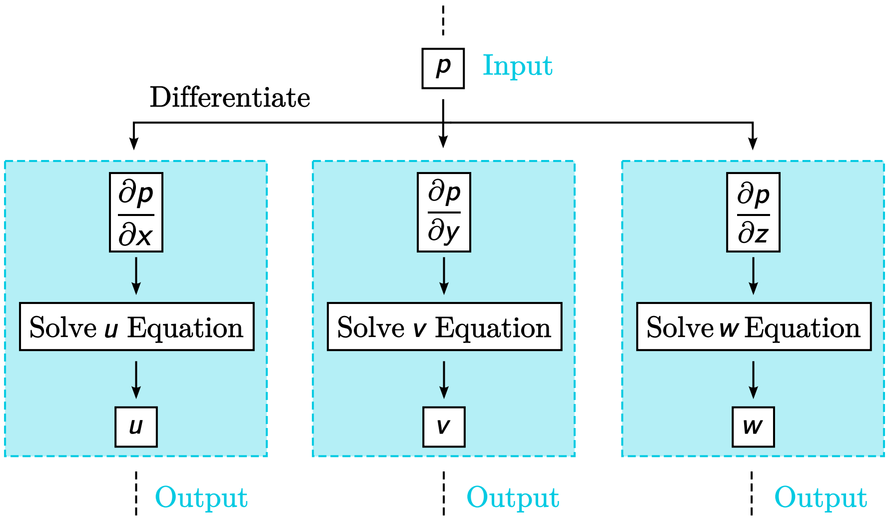
Output u

1. Choose an input pressure gradient, $\partial p/\partial x$

2. Solve the u momentum equation with this pressure gradient, to calculate u (this would be the velocity profile in the Hagen-Poiseuille flow)

> You can think of the velocity field as being generated by the pressure gradient. Different pressure gradients generate different velocity fields.

We can use the same approach to solve the v and w momentum equations. However, there is a key difference:



For the pressure gradients $\partial p/\partial x$, $\partial p/\partial y$ and $\partial p/\partial z$ to be consistent, it is easier to start with an input pressure field p and use numerical differentiation (covered in a future lecture) to calculate the pressure gradients in each direction.

3 The Continuity Equation

At this stage, we have chosen an input pressure field p and solved the u, v and w momentum equations to calculate the components of the velocity vector (u, v, w). The components of this velocity vector satisfy conservation of momentum with the input pressure field. However, we do not yet know whether they satisfy:

Fortunately this is relatively easy to check. Recall (from Lecture 1) that the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

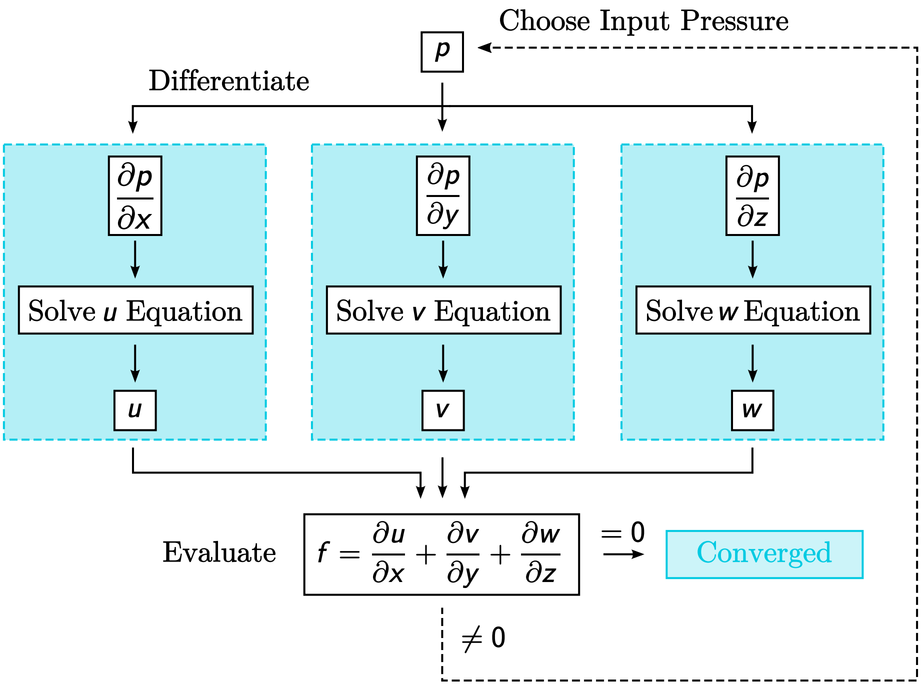
Define a new function f (the mass error):

$$f = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{3}$$

Substitute u, v and w into this equation.

- If $f = 0$ then the continuity equation is satisfied. ☐ True ☐ False
- If $f = 0$ then both the continuity and Navier-Stokes equations are satisfied. The algorithm is converged. ☐ True ☐ False
- If $f \neq 0$ then the Navier-Stokes equations are satisfied but the continuity equation is not. The algorithm is not converged. ☐ True ☐ False

The continuity equation acts as a check on the computed velocity field. If it is not satisfied, then the input pressure field that generated this velocity field is incorrect. We need to choose a different input pressure field.



> The central idea in the conceptual algorithm (and the SIMPLE algorithm) is to keep choosing different input pressure fields until the computed velocity field satisfies both the continuity and Navier-Stokes equations (resulting in $f = 0$).

So how do we choose the next input pressure field? The process of choosing the next input pressure field is described in the next lecture.

4 Notes

References

Articles

[2] S. Sutera and R. Skalak. 'The History of Poiseuille's Law'. In: *Annual Review of Fluid Mechanics* 25.1 (1993), pages 1–20. DOI: `10.1146/annurev.fl.25.010193.000245` (cited on page 1).

Books

[1] S. Patankar. *Numerical Heat Transfer and Fluid Flow.* Hemisphere Publishing Corporation, 1980. ISBN: 978-0891165224.

Version Control

This page keeps a record of updates to the content of the lecture notes. A semantic three-part numbering system (`major.minor.patch`) is used to refer to different versions of the lecture notes. For example, version 2.1.0 indicates major release 2, minor release 1, patch 0. Major releases indicate major / breaking changes to the content, minor changes indicate changes to individual equations or notation and patch numbers indicate formatting and graphical changes. If you identify any errors, please raise them in the comment section, so that the errors can be corrected and future readers be made aware of the corrections.

Version 1.0.0 (22nd August 2022) Initial Release.